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Integral Conditions for the Pressure in the Computation of Incompressible Viscous Flows

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The problem of finding the correct conditions for the pressure in the time discretized Navier–Stokes equations when the incompressibility constraint is replaced by a Poisson equation for the pressure is critically examined. It is shown that the pressure conditions required in a nonfractional-step scheme to formulate the problem as a system of split second-order equations are of an integral character and similar to the previously discovered integral conditions for the vorticity. The novel integral conditions for the pressure are used to derive a finite element method which is very similar to that developed by Glowinski and Pironneau and is the finite element counterpart of the influence matrix method of Kleiser and Schumann.

1. INTRODUCTION

The numerical simulation of incompressible viscous flows by means of a split formulation using only second-order partial differential equations poses a major difficulty due to the lack of boundary conditions for the vorticity or pressure variables (see, e.g., [1]). For the case of the nonprimitive variable equations, the problem is solved satisfactorily by supplementing the vorticity transport equation with conditions of an integral character [2–5], so that numerical solutions of the equations in split form are obtained by means of finite differences [5–7], finite elements [8, 9] and spectral methods [10, 11]. For the case of the primitive variables, the problem of finding the appropriate conditions, which supplement the Poisson equation for the pressure used to replace the incompressibility condition, is more difficult. Its solution depends on whether a fractional-step or a one-step method is employed to discretize the Navier–Stokes equations in time.

In the fractional-step or projection method developed by Chorin [12] and Temam [13] the momentum equation and the incompressibility condition are

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treated separately in two fractional steps. An intermediate velocity field which does not satisfy the condition of incompressibility is calculated from the time discretized momentum equation with the pressure term omitted. Such a velocity field is then decomposed into its divergenceless and irrotational components, the former being the correct final velocity field and the latter being proportional to the gradient of the pressure field. When the second step equations are formulated as a Poisson problem for the pressure, the appropriate boundary conditions for this variable are obtained quite naturally in the form of homogeneous [14, p. 399] or non-homogeneous [14, p. 414] Neumann conditions, depending on the degree of implicitness of the time discretization scheme used in the first step (see also [1, pp. 164–165]). However, the Neumann conditions for the pressure are derived after the time splitting process so that they are of a purely numerical character and are not satisfied by the exact pressure solution (cfr. [14, p. 399] or [1, p. 162]). Furthermore, fractional-step methods using an explicit treatment of the viscosity term present the difficulty that the velocity boundary conditions for the tangential components cannot be satisfied by the end-of-step velocity (see [15] for a finite element implementation of the method and [16, pp. 146–147] for a formulation of spectral type). The difficulty is related to the orthogonal decomposition theorem of the space of vector fields [17], which is basic to making the velocity field solenoidal in the second-step equations and assigns a very different role to the normal component of the velocity on the boundary with respect to the tangential ones. From a physical point of view, the incompressibility step is basically inviscid so that only the normal flow can be specified at the boundary in the second step [16].

In nonfractional-step methods the momentum equation and the continuity equation are to be satisfied at the same time so that there is no impediment to imposing the complete boundary condition for the velocity simultaneously with the incompressibility condition or any of its equivalent substitutes. However, in one-step methods the lack of boundary values for the pressure presents particularly severe difficulties. Only two correct methods using an exact and noniterative evaluation of the pressure at the boundaries in the Stokes problem have appeared so far. Glowinski and Pironneau proposed a finite element formulation for solving the incompressible Navier–Stokes equations in two and three dimensions [18] which resorts to an additional equation for a scalar velocity potential (see also [19, 20]). Kleiser and Schumann developed a formulation based on the influence matrix method [21] which has been employed to compute 3D channel flows [21] and 2D natural convection flows [22] by spectral approximations and 2D entry flows by finite differences [23]. These studies provide a correct perspective to the problem by introducing a supplementary (linear) problem to determine the lacking boundary values of the pressure. However, this fundamental concept has not received the general acceptance it deserves, maybe because the physical interpretation of the aforementioned methods is lacking.

This paper clarifies the problem in the spatial continuum for the case of the time discretized equations by providing the correct form of the conditions which supplement the Poisson equation for the pressure. The proper conditioning for the pressure (gradient) will be shown to be of an integral character and similar to that for the vorticity [5]. Using these conditions, a finite element formulation for solving steady and unsteady incompressible viscous flows is developed which is the natural counterpart for the primitive variables of the Glowinski–Pironneau method for the biharmonic problem [4]. Furthermore, the explicit recognition of the integral form of the pressure conditioning provides the interpretation of the Kleiser–Schumann influence matrix method [21] in the spatial continuum as well as a general framework for employing spatial discretizations of arbitrary type.

2. Problem Formulation

Kleiser and Schumann have shown that, in order to formulate the Navier–Stokes equations using a Poisson equation for the pressure in place of the incompressibility condition for the velocity, such a condition is to be retained on the boundary [21]. For the time discretized problem, using a one-step, two-level scheme with the viscous and pressure terms evaluated implicitly, and the advection term explicitly, the following system of equations is obtained:

$$(\nabla^2 - \gamma)\,\mathbf{u}^{n+1} - \nabla p^{n+1} = -\gamma \mathbf{u}^n + \mathbf{f}^n, \tag{1.a}$$

$$\nabla^2 p^{n+1} = -\nabla \cdot \mathbf{f}^n, \tag{1.b}$$

$$\nabla \cdot \mathbf{u}^{n+1}|_\Gamma = 0, \tag{1.c}$$

$$\mathbf{u}^{n+1}|_\Gamma = \mathbf{a}^{n+1}. \tag{1.d}$$

Here, $\mathbf{u}$ is the dimensionless velocity and p is the dimensionless pressure multiplied by the Reynolds number, called pressure for simplicity. Furthermore, $\gamma = \mathrm{Re}/\Delta t$, $\mathbf{f} = \mathrm{Re}(\mathbf{u} \cdot \nabla)\,\mathbf{u}$ and $\mathbf{a}$ is the velocity prescribed on the boundary Γ of the domain Ω and such that $\int \mathbf{n} \cdot \mathbf{a}\, d\Gamma = 0$. According to the above scheme, the solution of the time dependent Navier–Stokes equations is reduced to the solution of a sequence of "unsteady" Stokes problems.

Lack of boundary conditions for the pressure p^{n+1} prevents a direct solution of the properly formulated problem (1) as a system of split elliptic equations. However, proper conditions for p^{n+1} which make such a split formulation possible are obtained by considering the vector equivalent of the Green identity [24] for the Helmholtz operator $\nabla_\gamma^2 = (\nabla^2 - \gamma)$, namely,

$$\int (\mathbf{v} \cdot \nabla_\gamma^2 \mathbf{u} - \mathbf{u} \cdot \nabla_\gamma^2 \mathbf{v})\, d\Omega$$

$$= \int (\mathbf{n} \cdot \mathbf{v}\, \nabla \cdot \mathbf{u} - \mathbf{n} \cdot \mathbf{u}\, \nabla \cdot \mathbf{v} + \mathbf{n} \times \mathbf{v} \cdot \nabla \times \mathbf{u} - \mathbf{n} \times \mathbf{u} \cdot \nabla \times \mathbf{v})\, d\Gamma, \tag{2}$$

where $\mathbf{n}$ denotes the outward normal unit vector on Γ. The Green identity (2) implies that the pressure field p^{n+1} defined by problem (1) satisfies the following integral conditions:

$$\int (\nabla p^{n+1} - \gamma \mathbf{u}^n + \mathbf{f}^n) \cdot \boldsymbol{\eta}_\gamma \, d\Omega = -\int (\mathbf{n} \cdot \mathbf{a}^{n+1} \, \nabla \cdot \boldsymbol{\eta}_\gamma + \mathbf{n} \times \mathbf{a}^{n+1} \cdot \nabla \times \boldsymbol{\eta}_\gamma) \, d\Gamma \tag{3}$$

for any vector field $\boldsymbol{\eta}_\gamma$ solution of the Helmholtz problem

$$(\nabla^2 - \gamma) \, \boldsymbol{\eta}_\gamma = 0, \qquad \mathbf{n} \cdot \boldsymbol{\eta}_\gamma |_\Gamma \neq 0, \qquad \mathbf{n} \times \boldsymbol{\eta}_\gamma |_\Gamma = 0. \tag{4}$$

Due to the explicit treatment of the advection in the above scheme, the integral conditions (3) for p^{n+1} depend only on the data $\mathbf{f}^n$, $\mathbf{u}^n$ and $\mathbf{a}^{n+1}$ of problem (1) and do not involve values of the unknown velocity field $\mathbf{u}^{n+1}$ in the interior of the domain Ω. Since the number of linearly independent fields $\boldsymbol{\eta}_\gamma$ solution of problem (4) is equal to that of the boundary points, problem (1) can be reformulated in the following equivalent form:

$$\nabla^2 p^{n+1} = -\nabla \cdot \mathbf{f}^n, \tag{5a}$$

$$\int \nabla p^{n+1} \cdot \boldsymbol{\eta}_\gamma d\Omega = \int (\gamma \mathbf{u}^n - \mathbf{f}^n) \cdot \boldsymbol{\eta}_\gamma \, d\Omega - \int (\mathbf{n} \cdot \mathbf{a}^{n+1} \nabla \cdot \boldsymbol{\eta}_\gamma + \mathbf{n} \times \mathbf{a}^{n+1} \cdot \nabla \times \boldsymbol{\eta}_\gamma) d\Gamma; \tag{5b}$$

$$(\nabla^2 - \gamma) \, \mathbf{u}^{n+1} = \nabla p^{n+1} - \gamma \mathbf{u}^n + \mathbf{f}^n, \tag{6a}$$

$$\mathbf{u}^{n+1} |_\Gamma = \mathbf{a}^{n+1}. \tag{6b}$$

Equations (5)–(6) are the general factorized or split form of the time discretized Navier–Stokes equations for the primitive variables with the proper conditioning for the gradient of the pressure in the spatial continuum. Any method of spatial discretization of equations (5)–(6) is now appropriate.

There is an exchange of roles which occurs in the case of the time discretized equations when passing from the nonprimitive variable representation to the primitive variable one. By considering the simple case with homogeneous integral conditions for both the vorticity and the pressure, the equations for evaluating the vorticity [5] and the pressure at a given time have the following forms:

$$(\nabla^2 - \gamma) \, \zeta_\gamma = g, \qquad \int \zeta_\gamma \eta \, d\Omega = 0, \qquad \nabla^2 \eta = 0,$$

and

$$\nabla^2 p = h, \qquad \int \nabla p \cdot \boldsymbol{\eta}_\gamma \, d\Omega = 0, \qquad (\nabla^2 - \gamma) \, \boldsymbol{\eta}_\gamma = 0,$$

respectively. Note that the vorticity field, a solution of a Helmholtz equation, is orthogonal to the kernel of the Laplace operator, whereas the pressure field, a solution of a (nonhomogeneous) Laplace equation, is such that its gradient is orthogonal to the kernel of the same Helmholtz operator.

3. Pressure Equation with Integral Conditions

Now consider the solution of the nonstandard part of problem (5)–(6), namely, the Poisson equation (5a) for the pressure supplemented by the integral conditions (5b). To obtain the sought pressure field p^{n+1} separate it into its harmonic and nonharmonic components as follows:

$$p^{n+1}(\mathbf{x}) = p_0(\mathbf{x}) + \int p'(\mathbf{x}; s')\, \lambda^{n+1}(s')\, ds', \tag{7}$$

where, for any $s' \in \Gamma$,

$$\nabla^2 p' = 0, \qquad p'|_\Gamma = \delta(s - s'), \tag{8}$$

and

$$\nabla^2 p_0 = -\nabla \cdot \mathbf{f}^n, \qquad p_0|_\Gamma = \text{arbitrary}, \tag{9}$$

δ denoting the Dirac delta function over the boundary Γ. The unknown λ^{n+1} is determined by solving the linear problem obtained by imposing that p^{n+1} satisfies the integral condition (5b). This gives

$$\int A(s, s')\, \lambda^{n+1}(s')\, ds' = \beta^{n+1}(s) \tag{10}$$

where

$$A(s, s') = \int \nabla p' \cdot \boldsymbol{\eta}_\gamma \, d\Omega \tag{11}$$

and

$$\beta^{n+1}(s) = -\int \nabla p_0 \cdot \boldsymbol{\eta}_\gamma \, d\Omega + \int (\gamma \mathbf{u}^n - \mathbf{f}^n) \cdot \boldsymbol{\eta}_\gamma \, d\Omega$$
$$- \int (\mathbf{n} \cdot \mathbf{a}^{n+1}\, \nabla \cdot \boldsymbol{\eta}_\gamma + \mathbf{n} \times \mathbf{a}^{n+1} \cdot \nabla \times \boldsymbol{\eta}_\gamma)\, d\Gamma. \tag{12}$$

The supplementary linear problem (10) determines the harmonic component of the pressure field. The dimensionality of such a problem is equal to that of the boundary domain Γ.

4. Computational Scheme

A and β^{n+1} are evaluated by means of the Glowinski–Pironneau method [4, 18] generalized to vector fields. The fields $\boldsymbol{\eta}_\gamma$ are substituted by the vector functions $\mathbf{w}$, where

$$\mathbf{w} \text{ arbitrary in } \Omega, \qquad \mathbf{n}\cdot\mathbf{w}|_\Gamma=\delta(\tilde{s}-s), \qquad \mathbf{n}\times\mathbf{w}|_\Gamma=0, \tag{13}$$

and $\tilde{s}\in\Gamma$. The following "cascade" of elliptic problems is generated:

$$\nabla^2 p'=0, \qquad p'|_\Gamma=\delta(s-s'); \tag{14}$$

$$(\nabla^2-\gamma)\,\mathbf{u}'=\nabla p', \qquad \mathbf{u}'|_\Gamma=0; \tag{15}$$

and

$$\nabla^2 p_0=-\nabla\cdot\mathbf{f}^n, \qquad p_0|_\Gamma=\text{arbitrary}; \tag{16}$$

$$(\nabla^2-\gamma)\,\mathbf{u}_0=\nabla p_0-\gamma\mathbf{u}^n+\mathbf{f}^n, \qquad \mathbf{u}_0|_\Gamma=\mathbf{a}^{n+1}. \tag{17}$$

Application of the vector Green identity (2) and integration by parts provide

$$A(s,s')=\int[(\nabla\cdot\mathbf{u}')(\nabla\cdot\mathbf{w})+(\nabla\times\mathbf{u}')\cdot(\nabla\times\mathbf{w})+(\nabla p')\cdot\mathbf{w}+\gamma\mathbf{u}'\cdot\mathbf{w}]\,d\Omega, \tag{18}$$

$$\beta^{n+1}(s)=-\int[(\nabla\cdot\mathbf{u}_0)(\nabla\cdot\mathbf{w})+(\nabla\times\mathbf{u}_0)\cdot(\nabla\times\mathbf{w})+(\nabla p_0)\cdot\mathbf{w}+\gamma\mathbf{u}_0\cdot\mathbf{w}]\,d\Omega$$
$$+\int(\gamma\mathbf{u}^n-\mathbf{f}^n)\cdot\mathbf{w}\,d\Omega. \tag{19}$$

The unknown λ^{n+1} having been determined by solving the linear problem (10), the sought pressure p^{n+1} and velocity $\mathbf{u}^{n+1}$ are obtained as solutions of the following elliptic problems

$$\nabla^2 p^{n+1}=-\nabla\cdot\mathbf{f}^n, \qquad p^{n+1}|_\Gamma=\lambda^{n+1}+p_0|_\Gamma; \tag{20}$$

$$(\nabla^2-\gamma)\,\mathbf{u}^{n+1}=\nabla p^{n+1}-\gamma\mathbf{u}^n+\mathbf{f}^n, \qquad \mathbf{u}^{n+1}|_\Gamma=\mathbf{a}^{n+1}. \tag{21}$$

The use of the vector functions $\mathbf{w}$ instead of the vector fields $\boldsymbol{\eta}_\gamma$ offers two advantages: (i) the solution of Eqs. (4) and the storage of the fields $\boldsymbol{\eta}_\gamma$ are avoided; (ii) the arbitrariness of the functions $\mathbf{w}$ at all internal points of Ω can be exploited by choosing $\mathbf{w}=0$ inside Ω so that the domain of integration in relations (18)–(19) can be reduced to a narrow strip along the boundary Γ. However, the use of the functions $\mathbf{w}$ in place of the fields $\boldsymbol{\eta}_\gamma$ requires the solution of an additional vector elliptic problem for each scalar elliptic problem.

The present approach is different from the Glowinski–Pironneau method for the primitive variables [18] in three respects: (i) there is a single scalar elliptic problem, instead of two, for each vector elliptic problem to be solved; (ii) the auxiliary field $\mathbf{w}$ in Eq. (18)–(19) is a vector variable instead of a scalar one; (iii) the operator A of the supplementary linear problem (10) is not symmetric.

5. Applications

The elliptic problems introduced in Section 4 have been recast in weak variational form [25] and discretized in space by means of isoparametric quadrilateral bilinear elements with full integration, using the same mesh and the same order of interpolation for both the pressure and the velocity variables.

Consider the classical driven cavity problem [26]. In the Fig. 1 the transient and steady-state profiles of the horizontal velocity along the vertical centreline for $\mathrm{Re} = 100$ calculated by the present (primitive variable) method are compared with the corresponding results obtained by FD or FE nonprimitive variable methods. The pressure field is calculated by the present method without encountering the splitting in two uncoupled networks of pressure points (the so-called chequerboarding) or spurious pressure modes, even when a uniform mesh of equal elements is employed. Since the pressure is defined up to an arbitrary additive constant in this problem, the operator A defined in Eq. (11) turns out to be singular and the pressure value at a single point of the mesh is prescribed by standard methods.

The general validity of the present finite element formulation is verified using the half-channel problem proposed by Roache [27], which is also the Prototype of

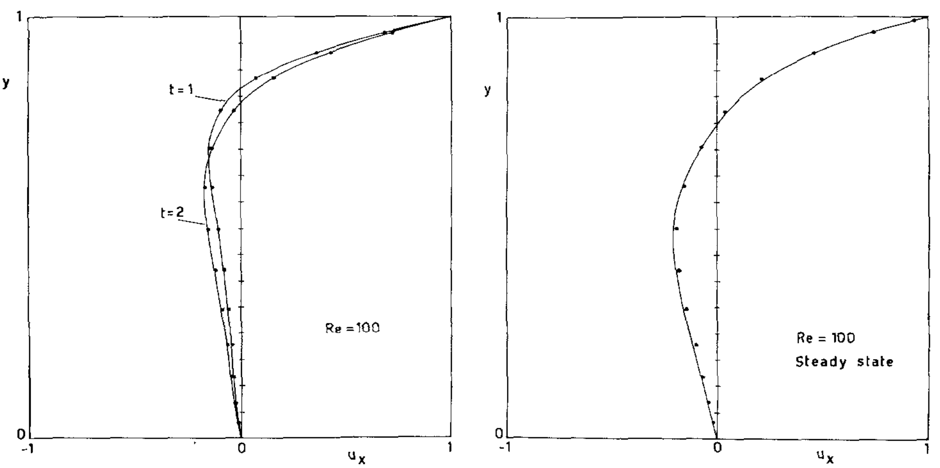

FIG. 1. Horizontal velocity along the vertical centreline in the driven cavity problem for $\mathrm{Re} = 100$. ● $\mathbf{u} - p$ present formulation, finite elements 16×16; ——, $\zeta - \psi$ formulation, finite differences 32×32 [6] or finite elements 16×16 [9].

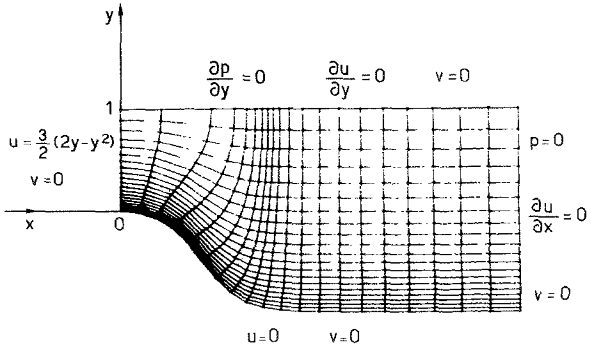

FIG. 2. Channel flow geometry, boundary conditions and mesh for Re = 10.

Flow in Complex Geometries of the IAHR Working Group on Refined Modelling of Flows (Fifth Meeting of IAHR, June 24–25 1982, Rome) [28]. The geometry, the boundary conditions and the finite element mesh for the case Re = 10 are shown in Fig. 2.

Numerical solutions obtained by the present method for Re = 10 and Re = 100 are in good agreement with those obtained by a fourth-order accurate spline ADI technique or bilinear finite elements using a $\zeta-\psi$ formulation and the same mesh. In both cases the points of separation and reattachment are located between the same grid points by the three methods. For the higher value of the Reynolds number (Fig. 3) the vorticity values along the channel wall calculated by the present method are compared with the results obtained by Roache [27].

These numerical results confirm the validity of the present formulation of the Navier–Stokes equations and its finite element implementation. The practical application of the proposed approach to large-scale real-life computations requires further efforts toward improving the overall computational efficiency of the method.

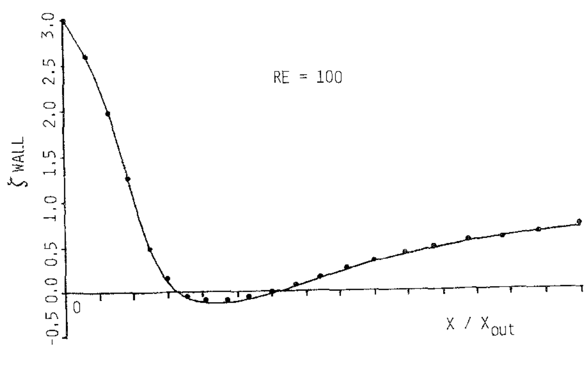

FIG. 3. Wall vorticity of the channel flow problem for Re = 100. ●, $\mathbf{u}-p$ finite element present formulation; ——, $\zeta-\psi$ finite difference formulation [27].

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